A Generalization of Pricing Options with Discrete Dividends in Markets with Daily Price Limits

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ABSTRACT

This paper proposes solutions for pricing options on stocks which pay discrete dividends in markets with daily price limits. We first extend the intraday density function of Guo and Chang (2020) to a multi-day one and use the framework of Haug et al. (2003) to value European options on stocks paying discrete dividends. Next, we adopt the fast Fourier transform (FFT) to derive accurate and efficient formulae for American options and, further, employ the three-point Richardson extrapolation to accelerate the computation. Finally, the accuracy of our proposed methods is verified by simulations.

Keywords: Daily Price Limit, Discrete Dividend, Early Exercise, Fast Fourier Transform, Multi-day Density Function, Richardson Extrapolation

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I. Introduction

This paper considers the problem of pricing options on stocks which pay discrete dividends in markets with daily price limits. Stocks frequently pay dividends, and many studies attempt to introduce more realistic assumptions about dividends after Black and Scholes (1973) derived option pricing formulae for non-dividend-paying stocks.[1] Brav et al. (2005) point out that firms provide dividends in discrete rather than continuous flow and CEOs are reluctant to change the size of dividends in order to maintain the investors' confidence. Although the discrete-payment setting could be more realistic than the continuous one, it gives rise to significant mathematical difficulty in pricing options due to jumps caused by discrete payments, even if they are constant (Dai and Chiu, 2014). In addition, Kim and Park (2010) point out that 23 out of 43 of the most important world markets use daily price limits. As per Guo and Chang's (2020) assertion, knowledge of pricing options in markets with daily price limits is quite limited, and our understanding of price limit mechanisms primarily comes from empirical studies. This illustrates the importance and contribution of this paper: it derives solutions for these types of pricing situations.

Black (1975) was the first to investigate the pricing problem of options on stocks paying a fixed dividend discretely. To incorporate discrete constant dividends into

[1] For example, Merton (1973) extends the Black-Scholes formulae by assuming continuous and constant dividend yields. Under similar assumptions of dividend yields, Kim (1990), Carr et al. (1992), and Jamshidian (1992) also provide pricing formulae for American options in an implicit form.

European option pricing, he suggests using the stock price minus the present value of dividends instead of the stock price itself. If firms escrow the dividend's value and run the business with the rest, the proposed method of Black (1975) will provide the exact value of European options. This approach often undervalues call options, and the mispricing becomes larger as the dividend is paid later in the option's lifetime.[2] Haug et al. (2003) indicate that the approximation suggested by Black (1975) for American options suffers from the same problem, as does the Roll-Geske-Whaley (RGW) model (Roll, 1977; Geske, 1979, 1981; Whaley, 1981). The RGW model applies a compound option approach to American options with similar approximations of stock price processes. Let S denote the stock price, D denote the size of the cash dividend to be paid at time t_D, σ denote the volatility, and r denote the risk-free interest rate. Many studies often solve this problem based on adjustments of volatility in combination with the escrowed dividend adjustment.[3] The method with $\hat{S} = S -$

[2] The price volatility before the ex-dividend could be too small with Black's approach because of the lower stock prices while the ex-dividend price process is fitted into a geometric Brownian motion (GBM) as the BSM formula.

[3] We shortly discuss some such approaches, all of which assume that the stock price can be described by a GBM: (1) An adjustment popular among practitioners is to replace the volatility σ with $\hat{\sigma} = (\sigma S)/(S - De^{-rt_D})$ (Chriss, 1997). Haug et al. (2003) show that this approach increases the volatility relative to the basic escrowed dividend process but the adjustment yields too high volatility if the dividend is paid out early in the option's lifetime. (2) A more sophisticated volatility adjustment replaces σ with $\sigma_2 = (\sigma S)/(S - De^{-rt_D})$ as before, but not for the entire lifetime of the option (Haug and Haug, 1998; Beneder and Vorst, 2001). The idea behind the approximation is to leave volatility unchanged in the time before the dividend payment and to apply the volatility σ_2 after the dividend payment. Because the BSM model requires one volatility as input, some sort of weight must be assigned to each of σ and σ_2. The single input volatility is then computed as

De^{-rt_D} , $\sigma_2 = (\sigma S)/(S - De^{-rt_D})$, and $\hat{\sigma} = \sqrt{\left(\sigma^2 t_D + \sigma_2^2 (T - t_D)\right)/T}$ may be referred to the spot volatility adjusted approximation. Buryak and Guo (2012) introduce a different approximation based on the set-up of Bender and Vorst (2001). They modify the strike price from K to $\hat{K}$ by setting $\hat{K} = K + De^{r(T-t_D)}$ and adjust the volatility σ to $\hat{\sigma} = \sqrt{\left(\sigma^2 t_D + \sigma_2^2 (T - t_D)\right)/T}$ where $\sigma_2 = \sigma S/(S + De^{-rt_D})$. The method of Buryak and Guo (2012) may be referred to as the strike volatility adjusted approximation. Buryak and Guo (2012) take the hybrid approximation ($\hat{S} = S - D_S$ and $\hat{K} = K + D_K$ where $D_S = De^{-rt_D}(T - t_D)/T$ and $D_K = De^{-rt_D} t_D/T$) above as a starting point but then adjust the volatilities in a manner related to the volatility adjustment schemes mentioned earlier with $\sigma_S = \sigma S/(S - D_S)$, $\sigma_K = \sigma S/(S + D_K)$, $\bar{\sigma}_S = \sqrt{\left(\sigma_S^2 t_D + \sigma^2 (T - t_D)\right)/T}$, $\bar{\sigma}_D = \sqrt{\left(\sigma^2 t_D + \sigma_K^2 (T - t_D)\right)/T}$, and $\hat{\sigma} = \sqrt{\bar{\sigma}_S \bar{\sigma}_D}$. This method may be referred to the hybrid volatility approximation. Haug et al. (2003) claim that, even if this approach

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$\hat{\sigma} = \sqrt{\left(\sigma^2 t_D + \sigma_2^2 (T - t_D)\right)/T}$ where T is the time of expiration for the option. This is still simply an adjustment to parameters of the GBM price process that ensures the adjusted price process remains a GBM at odds with the true ex-dividend price process. Haug et al. (2003) show that this method performs particularly poorly in the presence of multiple dividends. (3) Bos et al. (2003) suggest a more sophisticated volatility adjustment to overcome the problems with the escrowed dividend price process. Numerical results of Haug et al. (2003) show that this approach performs poorly for very high dividends and seemingly also performs poorly for long-term options with multiple dividends. (4) A slightly different way to implement the escrowed dividend process is to adjust the stock price and strike (Bos and Vandermark, 2002). In contrast with the spot volatility approximation and the strike volatility approximation, this method does not adjust the volatility and may be referred to the hybrid approximation.

seems to work better than the approximations mentioned above, it still suffers from approximation errors for large dividends.[4]

Haug et al. (2003) point out several problems and weaknesses of the current approaches mentioned above: (a) logical flaws. The logical flaw of an escrowed dividend process is that the resulting stock price process changes with the option expiration. Whatever the stock price process is, it cannot depend upon which option you happen to be considering. (b) Ill-defined stock price processes. Wilmott et al. (1993) suggests letting the company go bankrupt if the dividend is larger than the asset price as this approach avoids negative stock prices. (c) Arbitrage issues: this is illustrated by an example in Haug et al. (2003) which notes that the arbitrage occurs because the RGW model is mis-specified in that the dynamics of the stock price process depends on the timing of the dividend. Similar examples have been discussed by Bender and Vorst (2001) and Frishling (2002).

In addition to the arguments of Haug et al. (2003), we believe that the discussion of the impacts of discrete dividends on options could not be limited in the case of the geometric Brownian motion. We extend the method of Haug et al. (2003) by studying

[4] An alternative to the escrowed dividend approximation is to use the non-recombining lattice method (Hull, 2000). If implemented as a binomial tree, one builds a new tree from a node on each dividend payment date. A problem with all non-recombining lattices is that they are time consuming to evaluate. Schroder (1988) describes how to implement discrete dividends in a recombining tree. However, this approach is based on the escrowed dividend process, and could significantly misprice options. Wilmott et al. (1993) indicates what seems to be a sounder approach to ensure a recombining tree for the spot price process with a discrete dividend.

daily price limits. Such an extension is important because most stock markets around the world use price limits. Price limits are believed to mitigate excessive price volatility, lower panic behavior, and/or minimize price manipulation.[5] Despite their significant presence, however, impacts of these price limit mechanisms on options are not well understood, and there remain many unanswered questions about how to make early-exercise decisions regarding market regulation because of the lack of appropriate study tools. In this paper, we extend the intraday density function of Guo and Chang (2020) to a multi-day density function. Then, we use the framework of Haug et al. (2003) to value European options on stocks with discrete dividends in daily price limit markets. As for American options, we derive an efficient formula for the early exercise premium and employ the three-point Richardson extrapolation to accelerate the computation. In addition, we also explore the influence of dividends, price limits, and interest rates on the decision of early exercise.

The rest of this paper is organized as follows: in Section 2, the model and methodology are briefly described. Section 3 provides a comparison of our proposed solution with simulations and illustrates our findings. Section 4 presents the conclusion.

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[5] See Kim and Rhee (1997), Kim (2001), Kim and Yang (2004), and Kim and Park (2010).

II. Model and Methodology

2.1 Framework of Haug et al. (2003)

We first consider the framework of Haug et al. (2003) to value a European or American option on a stock that pays a discrete dividend at time $t = t_D$. In this framework, let $M_t = M(t)$ denote the so-called cum dividend process, which is when any dividends are reinvested immediately back into the security. In general, M_t is not the market price of the stock, but instead is the market price of a hypothetical mutual fund that only invests in that stock. Let S_t denote the market price of the stock at time t. Sometimes, we will call S_t the ex-dividend process. If there are no dividends, then $S_t = M_t$ for all t. Even if the company pays a dividend, we can always arrange things so that $S_0 = M_0$, which guarantees (by the law of one price) that $S_t = M_t$ for all $t < t_D$.

Now we consider an unprotected Euro-style option for stocks issued by a company that declares a single discrete dividend of size D and the ex-dividend date t_D during the option holding period. If the company pays a dividend D, the stock price at the ex-dividend date must drop by the same amount: $S(t_D) = S(t_D^-) - D = M(t_D^-) - D$. The notation of t_D^- is the time instantaneously before the ex-dividend date t_D. Because the stock price represents the price of a limited liability security, we must have $S(t_D) \geq 0$; so, there is a fundamental contradiction between these last two concepts if $M(t_D^-) < D$. We resolve it by the following minimal modification to the dividend policy. We assume that the company will indeed pay out its declared amount if $M(t_D^-) > D$, abbreviating $M^- = M(t_D^-)$. However, in the case where $M^- < D$, we assume that the company pays some lesser amount $\Delta(M^-)$ whereby $0 \leq \Delta(M^-) \leq M^-$. There are two natural dividend policy choices, namely $\Delta(M^-) = M^-$ (liquidator) and $\Delta(M^-) = 0$ (survivor). The first case allows liquidation because the ex-dividend stock price would be absorbed at zero. The second case allows survival because the stock price process can then attain strictly positive values after the dividend payment. Thus, the actual dividend paid becomes the random variable $\mathcal{D}(M)$, where

$$\mathcal{D}(M) = \begin{cases} D, & \text{if } M > D \\ \Delta(M) \leq M, & \text{if } M \leq D \end{cases}. \tag{1}$$

In (1), D is the declared dividend—a constant, independent of M. The functional form for $\mathcal{D}(M)$ is any function that preserves limited liability. Then, the market price of the security evolves, using GBM as the prototype of the cum-dividend process ($dM_t = rM_t dt + \sigma M_t dB_t$), as follows:

$$dS_t = \left[rS_t - \delta(t - t_D)\mathcal{D}\left(S_{t_D^-}\right)\right]dt + \sigma S_t dB_t \tag{2}$$

where $\delta(t - t_D)$ is Dirac's delta function centered at t_D, σ is a constant volatility, and B is a standard Brownian motion. Note that $S_t = M_t$ for all $t < t_D$ and $S_{t_D} = M_{t_D} - \mathcal{D}(M_{t_D})$. For $t > t_D$ (post ex-dividend date), little can be said about S_t

given only knowledge of M_t (all we can say is that $S_t < M_t$ if $D > 0$).[6] Let $\phi(M_0, M_t, t)$ denote the cum-dividend transition density, which is the probability density for an initial state M_0 to evolve to the final state M_t over a given time t. For GBM, $\phi(M_0, M_t, t)$ is the familiar log-normal density.

We write $V_E(M_t, t; D, t_D)$ for the time-t fair value of a European-style option that expires at time T in the presence of a discrete dividend D paid at time t_D. The last two arguments are the main parameters in the fully specified dividend policy $\{t_D, \mathcal{D}(M)\}$ where $t < t_D < T$. If there is no dividend between time t and the option expiration T, we simply drop the last two arguments and write $V_E(M_t, t)$, which is a well-known formula in the absence of dividends. For simplicity, the strike price K, option expiration T, and other parameters and state variables have been suppressed. We can therefore obtain the option price by discounting the expected value at t_D, as follows:

Remark. Integral form of a European-style option price (Haug et al., 2003)

$$V_E(M_0, 0; D, t_D) = e^{-rt_D} \int_0^{\infty} V_E(M - \mathcal{D}(M), t_D)\, \phi(M_0, M, t_D) dM. \tag{3}$$

[6] One may assume that, after the ex-dividend date, the stock price (S_t, for $t > t_D$) still can be approximated by another GBM process. This assumption constitutes the so-called piecewise geometric Brownian motion process. For instance, Dai and Chiu (2014) assume that the stock price process prior to time t_D and after time t_D can be separately modelled by two different lognormal-diffusive stock price processes.

Note that the adoption by the company of a single discrete dividend policy $\{t_D, \mathcal{D}(M)\}$ causes the fair value of a European-style option to change from $V_E(M_0, 0)$ to $V_E(M_0, 0; D, t_D)$. For the results of (3) to be useful, we need to be able to solve our model in the absence of dividends.

In the case of American call options, an optimal early exercise is limited to the ex-dividend date. Therefore, Haug et al. (2003) also provide pricing for American call options in an integral form, as follows:

Remark. Integral form of an American call option price (Haug et al., 2003)

$$C_A(M_0, 0; D, t_D) = e^{-rt_D} \int_0^{\infty} max\{(M-K)^+, C_E(M - \mathcal{D}(M), t_D)\} \, \phi(M_0, M, t_D) dM. \tag{4}$$

Early exercise is never optimal unless there is a finite solution of M^*, satisfying

$$M^* - K = C_E(M^* - D, t_D) \tag{5}$$

where we assume that $K > D$ (a virtual certainty in practice). We can break up the integral into pieces, as follows:

$$C_A(M_0, 0; D, t_D) =$$
$$e^{-rt_D} \int_0^{M^*} C_E(M - \mathcal{D}(M), t_D) \, \phi(M_0, M, t_D) dM + e^{-rt_D} \int_{M^*}^{\infty} (M - K) \, \phi(M_0, M, t_D) dM \, . \tag{6}$$

With the sequence of dividends $\{(D_i, t_i)\}_{i=1}^{n}$, $t_1 < t_2 < \cdots < t_n$, the argument leading to Eq. (3) still holds. This should be repeated iteratively, starting at time t_{n-1} by applying (3) to the last dividend (D_n, t_n). This procedure could involve evaluating a very time-consuming n-fold integral. Therefore, we need an efficient and accurate way to implement it.

For American put options, it can be optimal to exercise at any time prior to expiration, even in the absence of dividends. So, in this case, we are generally forced to a numerical solution; this is a well-known backward iteration. What may differ from what we are used to is that we must allow for an instantaneous drop of $\mathcal{D}(M)$ on the ex-date. The Richardson extrapolation technique is one possible solution to obtain an efficient scheme for American options on a stock paying discrete dividends. For example, according to Kim (1990) and Chang et al. (2016):

$$P_A\big(M(t) - \mathcal{D}(M_{t_D}), t\big)$$

$$= P_E\big(M(t) - \mathcal{D}(M_{t_D}), t\big) + rK \int_t^T e^{-r(s-t)} \left\{\int_0^{M^*(s)} \phi(M_t, M, s-t) dM\right\} ds \quad \text{for } t \geq t_D. \tag{7}$$

The critical exercise boundary solves the following integral equation for $M^*(t)$:

$$K - \Big(M^*(t) - \mathcal{D}\big(M_{t_D}\big)\Big)$$

$$= P_E\big(M^*(t) - \mathcal{D}\big(M_{t_D}\big), t\big) + rK \int_t^T e^{-r(s-t)} \left\{\int_0^{M^*(s)} \phi(M^*(t), M, s-t) dM\right\} ds$$

$$\text{for } T \geq t \geq t_D. \tag{8}$$

Once $M^*(t)$ is obtained, the price of the American put option can be calculated based on Eq. (7). Solving for $M^*(t)$ needs to be conducted recursively. We need to solve for $M^*(s)$ for $s \in (t, T]$. In order to be efficient so as to rapidly evaluate American options without approximating the whole early exercise boundary between t_D and T, we follow Huang et al. (1996) and Chang et al. (2016) to utilize a three-point Richardson extrapolation to accelerate the recursive integration method. The Richardson extrapolation scheme gains efficiency without sacrificing much accuracy. Our proposed model is implemented in a similar way. Assuming that the option can be respectively exercised only once, twice, or three times between t_D and T, and denoting the corresponding option prices as P_1, P_2, and P_3, the three-point Richardson extrapolation for the American put option could be expressed as follows:

$$\hat{P}_A = \frac{1}{2}(P_1 - 8P_2 + 9P_3) \tag{9}$$

where $\hat{P}_A$ denotes the approximated American put option value. In addition, we have

$$E_t^Q\{P_A(S_{t_D}, t_D; D, t_D)\} = \int_0^\infty P_A(M - \mathcal{D}(M), t_D)\, \phi(M_t, M, t_D - t) dM. \tag{10}$$

In the case of the liquidator dividend policy, Eq. (10) reduces to

$$E_t^Q\{P_A(S_{t_D}, t_D; D, t_D)\}$$
$$= P_A(0, t_D) \int_0^D \phi(M_t, M, t_D - t) dM + \int_D^\infty P_A(M - D, t_D)\, \phi(M_t, M, t_D - t) dM$$
$$= K \int_0^D \phi(M_t, M, t_D - t) dM + \int_D^\infty P_A(M - D, t_D)\, \phi(M_t, M, t_D - t) dM. \tag{11}$$

Let $P_A^E(M(t), t; D, t_D) \equiv e^{-r(t_D-t)} E_t^Q\{P_A(S_{t_D}, t_D; D, t_D)\}$. Then, for $0 \leq t < t_D$, we have

$$P_A(M(t), t; D, t_D)$$
$$= P_A^E(M(t), t; D, t_D) + rK \int_t^{t_D} e^{-r(s-t)} \left\{\int_0^{M^*(s)} \phi(M_t, M, s-t) dM\right\} ds. \tag{12}$$

The critical exercise boundary solves the following integral equation for $M^*(t)$:

$$K - M^*(t)$$
$$= P_A^E(M^*(t), t; D, t_D) + rK \int_t^{t_D} e^{-r(s-t)} \left\{\int_0^{M^*(s)} \phi(M^*(t), M, s-t) dM\right\} ds$$
$$\text{for } 0 \leq t < t_D. \tag{13}$$

Once $M^*(t)$ is obtained, the price of the American put option can be calculated based on Eq. (12). Again, solving for $M^*(t)$ needs to be conducted recursively. We need to solve for $M^*(s)$ for $s \in (t, t_D]$. To be efficient and rapidly evaluate American options without approximating the entire early exercise boundary between t and t_D, we can utilize a three-point Richardson extrapolation to accelerate the recursive integration method. Assuming that the option can be exercised only once, twice, or three times between t and t_D, and denoting the corresponding option prices as $\breve{P}_1$, $\breve{P}_2$, and $\breve{P}_3$, the three-point Richardson extrapolation for the American put option could be expressed as follows:

$$\breve{P}_A = \frac{1}{2}\left(\breve{P}_1 - 8\breve{P}_2 + 9\breve{P}_3\right) \tag{14}$$

where $\breve{P}_A$ denotes the approximated American put option value. Our approach may not be limited to the GBM price process, and M_t could follow a very general continuous-time stochastic process whose transition density is known.

2.2 Intraday characteristic function

We next extend this new efficient scheme to pricing European options on stocks which pay discrete dividends in markets with daily price limits. We first extend the intraday density function of Guo and Chang (2020) to a multi-day density function for stocks in markets with daily price limits. Consider an example of a European option with maturity T on stocks with daily price limits defined as follows: (A.1) price limits are determined by stock prices at date t_i, where $i = 0, \cdots, N$ and $t_0 = 0 < t_1 < t_2 < t_3 < \cdots < t_N = T$. The time interval between t_i and t_{i+1} is often one day. (A.2) In each time interval, the pricing process is a function of a geometric Brownian motion until price limits are reached. (A.3) After reaching a boundary, the stock price may remain on the boundary for a time or rebound away from the boundary. Hence, as Ban et al. (2000) claimed, the least complicated natural process in each time interval is given by the following stochastic differential equation:

$$\begin{cases} dS_t = \sigma S_t I_{(a,b)}(S_t) dW_t + \theta S_t I_{(a,b)}(S_t) dt + \delta_1 d\emptyset_t - \delta_2 d\varphi_t \\ I_{\{a\}} dt = \rho_1 d\emptyset_t \\ I_{\{b\}} dt = \rho_2 d\varphi_t, \end{cases} \tag{15}$$

where W_t denotes a standard Brownian motion, θ denotes the drift term, and ϕ

and φ are, respectively, local times at a (the lower bound) and b (the upper bound) under the physical measure. ρ is the viscosity of the boundary with $\rho \geq 0$; larger values of ρ could inhibit the change in the stock price. $\delta(\geq 0)$ denotes the elasticity of the boundary; as δ increases, the stock price rebounds more violently. This is Ban et al.'s (2000) intraday model of daily price limit markets. With the vanishing transaction cost technique, Ban et al. (2000) showed that the transaction cost vanishes sufficiently fast and the hedging error vanishes as the size of the discretization interval shrinks to zero. Therefore, they derived the following intraday partial differential equation (PDE) for the value of the contingent claim C with maturity T under the price-limit process described by Eq. (16):

$$\begin{cases} \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) + rS\frac{\partial C}{\partial S}(S,t) - rC(S,t) = 0 \\ C(S,T) = Y(S) \\ \frac{\partial C}{\partial t}(a,t) + ra\frac{\partial C}{\partial S}(a,t) - rC(a,t) = 0 \\ \frac{\partial C}{\partial t}(b,t) + rb\frac{\partial C}{\partial S}(b,t) - rC(b,t) = 0 \end{cases} \tag{16}$$

where $(S,t) \in [a,b] \times [0,T]$, r denotes the risk-free rate, and $Y(S)$ is the value of the contingent claim expired at the end of the day. Because the differential equations of Ban et al. (2000) are independent of risk preferences and local time terms, if risk preferences and local time terms do not enter the equations, they cannot affect their solution. A very simple assumption can be made that all investors are risk neutral and both boundaries (δ_1=δ_2=0) are absorptive. Therefore, the new process becomes dS_t/S_t=$\vartheta I_{(a,b)}(S_t)\,dt$+$\sigma I_{(a,b)}(S_t) d\widetilde{W}_t$, where ϑ denotes the drift parameter and $\widetilde{W}_t$

denotes a standard Brownian motion under the risk-neutral measure. The drift parameter ϑ can be determined with the requirement of retaining the Martingale property.[7] After defining Z_t=ln S_t and applying Itô's lemma, we have

$$dZ_t = \left(\vartheta - \frac{1}{2}\sigma^2\right) I_{(L,U)}(Z_t)dt + \sigma I_{(L,U)}(Z_t)d\widetilde{W}_t$$

$$\equiv \mu I_{(L,U)}(Z_t)dt + \sigma I_{(L,U)}(Z_t)d\widetilde{W}_t \tag{17}$$

where μ=ϑ-$\sigma^2/2$, and L $\{= ln(a) = ln[(1-\alpha)S_0]\}$ and U $\{= \ln(b) = \ln[(1+\beta)S_0]\}$ are the lower bound and upper bound of Z_t, respectively. If $\alpha = \beta = \gamma$, $[L, U]$ = $[Z_0 + \ln(1-\gamma), Z_0 + \ln(1+\gamma)]$. Thus, the intraday transition density $p(t, Z_0, Z_t)$ with $L < Z_t < U$ must satisfy the following backward equation[8]:

$$\begin{cases} \frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial p}{\partial x}, & t > 0, L < x < U, L < y < U \\ \lim_{x \to L} p(t, x, y) = 0, & t > 0, L < y < U \\ \lim_{x \to U} p(t, x, y) = 0, & t > 0, L < y < U \\ \lim_{t \to 0} p(t, x, y) = \delta(y - x), & L < x < U, L < y < U, \end{cases} \tag{18}$$

where $x = Z_0$, $y = Z_t$, and δ denotes the Dirac delta function. Substituting $x = \hat{x} + L$, $y = \hat{y} + L$, and $d = U - L$ into $p(t, x, y)$ yields $\hat{p}(t, \hat{x}, \hat{y})$, satisfying[9]

$$\begin{cases} \frac{\partial \hat{p}}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 \hat{p}}{\partial \hat{x}^2} + \mu \frac{\partial \hat{p}}{\partial \hat{x}}, & t > 0, 0 < \hat{x} < d, 0 < \hat{y} < d \\ \lim_{\hat{x} \to 0} \hat{p}(t, \hat{x}, \hat{y}) = 0, & t > 0, 0 < \hat{y} < d \\ \lim_{\hat{x} \to d} \hat{p}(t, \hat{x}, \hat{y}) = 0, & t > 0, 0 < \hat{y} < d \\ \lim_{t \to 0} \hat{p}(t, \hat{x}, \hat{y}) = \delta(\hat{y} - \hat{x}), & 0 < \hat{x} < d, 0 < \hat{y} < d. \end{cases} \tag{19}$$

Guo and Chang (2020) show that

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7 The measure that meets the requirement of $\tilde{E}[e^{-rT}S_T|S_0] = S_0$ is called the risk-neutral measure (see Kou and Wang, 2004).

8 Please refer to Bhattacharya and Waymire (1990) for details of the backward equation.

9 Note that $p(t, x, y) = \hat{p}(t, \hat{x}, \hat{y})$ with the specification of $x = \hat{x} + L$ and $y = \hat{y} + L$.

$$p(t,x,y) = \hat{p}(t,\hat{x},\hat{y}) = \hat{p}(t,x-L,y-L)$$

$$= \frac{2}{U-L} exp\left\{\frac{\mu(y-x)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right\} \sum_{m=1}^{\infty} exp\left\{-\frac{m^2\pi^2\sigma^2 t}{2(U-L)^2}\right\} \times sin\left(\frac{m\pi(x-L)}{U-L}\right) sin\left(\frac{m\pi(y-L)}{U-L}\right) \tag{20}$$

where $t > 0$, $L < x$, and $y < U$. In addition,

$$p(t,x,\{L\}) = \Pi_x(\tau_L < \tau_U) - \int_L^U \Pi_y(\tau_L < \tau_U) p(t,x,y) dy, \tag{21}$$

and

$$p(t,x,\{U\}) = \Pi_x(\tau_U < \tau_L) - \int_L^U \Pi_y(\tau_U < \tau_L) p(t,x,y) dy, \tag{22}$$

where τ_L and τ_U denote the stopping time at L and U, respectively. Given the initial position $Z_0 = x$, the expression $\Pi_x(\tau_L < \tau_U)$ and $\Pi_x(\tau_U < \tau_L)$ can be defined and given by[10]

$$\begin{cases} \Pi_x(\tau_L < \tau_U) = \frac{1-\exp(2\mu(U-x)/\sigma^2)}{1-\exp(2\mu(U-L)/\sigma^2)} \\ \Pi_x(\tau_U < \tau_L) = \frac{1-\exp(-2\mu(x-L)/\sigma^2)}{1-\exp(-2\mu(U-L)/\sigma^2)}. \end{cases} \tag{23}$$

Given the intraday transition density under the chosen measure, the intraday characteristic function can be further deduced. The characteristic function is defined by

$$J_1(\phi, Z_0, t_1) \equiv \tilde{E}\left[exp\left(i\phi Z_{t_1}\right)|Z_0\right]$$

$$= \int_L^U e^{i\phi y} p(t_1,x,y) dy + e^{i\phi L} p(t_1,x,\{L\}) + e^{i\phi U} p(t_1,x,\{U\}). \tag{24}$$

Note that $p(t_1,x,\{L\})$ and $p(t_1,x,\{U\})$ are constants because they depend only on ϑ, σ, α, and β. The characteristic function of the closing price is given by

[10] Please refer to Bhattacharya and Waymire (1990) for details of the proof.

$$J_1(\phi, Z_0, t_1)$$

$$= e^{i\phi Z_0}\{C \sum_{m=1}^{\infty} F_m G_m(\phi) + (1-\alpha)^{i\phi} p(t_1, x, \{L\}) + (1+\beta)^{i\phi} p(t_1, x, \{U\})\}, \quad (25)$$

where $C = \frac{2}{U-L} exp\left(\frac{-\mu^2 t_1}{2\sigma^2}\right) exp\left(\frac{\mu ln(1-\alpha)}{\sigma^2}\right)$, $F_m = exp\frac{-m^2\pi^2\sigma^2 t_1}{2d^2} sin\left(\frac{-m\pi\, ln(1-\alpha)}{U-L}\right)$, and $G_m(\phi) = \exp(i\phi \ln(1-\alpha)) \int_0^{U-L} exp\left(\frac{(i\phi\sigma^2+\mu)y}{\sigma^2}\right) \sin\left(\frac{m\pi y}{U-L}\right) dy$. Note that $J_1(\phi, Z_0, t_1)$ contains two parts, which are $e^{i\phi Z_0}$ and

$$H(\phi, t_1, \alpha, \beta) = C \sum_{m=1}^{\infty} F_m G_m(\phi) + (1-\alpha)^{i\phi} p(t_1, x, \{L\}) + (1+\beta)^{i\phi} p(t_1, x, \{U\}), \quad (26)$$

where $H(\phi, t_1, \alpha, \beta)$ is a function of ϕ without Z_0. Therefore, under the chosen measure, the multiday characteristic function of the logarithm price at the end of the N^{th} day is

$$J_N(\phi, Z_0, t_N) \equiv \tilde{E}\left[exp\left(i\phi Z_{t_N}\right)|Z_0\right] = e^{i\phi Z_0} H(\phi, t_1, \alpha, \beta)^N \quad (27)$$

where t_1 is the time period of one time interval (one day).

2.3 Pricing option using the fast Fourier transform (FFT)

Given the characteristic function of the logarithm price, Carr and Madan (1999) show that the call price can be obtained numerically using the inverse transform

$$C_T(k) = \frac{exp(-\hat{\alpha}k)}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) dv \quad (28)$$

for a range of positive values of $\hat{\alpha}$, where $k = log(K)$, and

$$\psi_T(v) = \frac{e^{-rT} J_{N+\eta}(v-(\hat{\alpha}+1)i, Z_0, T)}{\hat{\alpha}^2+\hat{\alpha}-v^2+i(2\hat{\alpha}+1)v}. \quad (29)$$

In order to avoid a highly oscillatory integrand in the Fourier inversion for out-of-the money options with very short maturities, Carr and Madan (1999) further suggest using

$$C_T(k) = \frac{1}{sinh(\hat{\alpha}k)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\gamma_T(v)dv \tag{30}$$

where $\gamma_T(v) = (\zeta_T(v - i\hat{\alpha}) - \zeta_T(v + i\hat{\alpha}))/2$ and

$$\zeta_T(v) = e^{-rT}\left(\frac{1}{1+iv} - \frac{e^{rT}}{iv} - \frac{J_{N+\eta}(v-i, Z_0, T)}{v^2 - iv}\right). \tag{31}$$

Hence, the approximation for $C_T(k)$ in Eq. (32) using the fast Fourier transform (FTT) is given by

$$C_T(k_u) = \frac{exp(-\hat{\alpha}k_u)}{\pi}\sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j}\psi(v_j)\frac{\xi}{3}\left[3 + (-1)^j - \varrho_{j-1}\right] \tag{32}$$

where $v_j = \xi(j-1)$, $k_u = -b + \lambda(u-1)$ for $u = 1,2,\cdots,M$, $b = M\lambda/2$, $\lambda\xi = 2\pi/M$, and ϱ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The use of the FFT for calculating out-of-the-money option prices is given by

$$C_T(k_u) = \frac{1}{\sinh(\hat{\alpha}k_u)}\frac{1}{\pi}\sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j}\gamma(v_j)\frac{\xi}{3}\left[3 + (-1)^j - \varrho_{j-1}\right]. \tag{33}$$

2.4 Multi-day density function

To derive the multi-day density function, we consider the following transformation

$$p_N(t_N, x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi y} J_N(\phi, Z_0, t_N)\, d\phi \tag{34}$$

where $x = Z_0$ and $y = Z_{t_N}$. Then, we apply our multi-day density function with the framework of Haug et al. (2003) to value a Euro-style equity option on a stock that pays a discrete dividend. We write $C_E(S_0, 0; D, t_D)$ for the time-0 fair value of a European call option that expires at time T in the presence of a discrete dividend D paid at time-t_D. According to Eq. (3), we have

$$C_E(S_0, 0; D, t_D) = e^{-rt_D} \int_0^\infty C_E(S_{t_D^-} - \mathcal{D}(S_{t_D^-}), t_D)\, p_N\big(t_D^-, lnS_0, lnS_{t_D^-}\big) \frac{1}{S_{t_D^-}} dS_{t_D^-}. \tag{35}$$

where $C_E(S_t, t)$ is the time-t price of a European call option expired at time T on a non-dividend paying stock. In the case of liquidator dividend policy, namely $D(M) = \Delta(M^-) = M^-$, we have

$$C_E(S_0, 0; D, t_D) = e^{-rt_D} \int_D^\infty C_E(S_{t_D^-} - D, t_D)\, p_N\big(t_D^-, lnS_0, lnS_{t_D^-}\big) \frac{1}{S_{t_D^-}} dS_{t_D^-}. \tag{36}$$

To calculate the integral of Eq. (36), we use the Riemann sum to approximate the integral as

$$e^{-r_{td}} \sum_{M=M_{min}}^{M=M_{max}} V_E(M - D(M), t_D) \phi(M_0, M, t_D) \Delta M \tag{37}$$

where the M_{min}(=max(D, $S_0(1-L)^N$))and M_{max}(=$S_0(1+U)^N$) are the minimum and maximum stock prices prior to the ex-dividend moment in markets with daily price limits.

2.5 Early exercise premium

As for American put options, it is difficult to find an analytical solution to the boundary and we focus on numerical solutions. With the Richardson three-point extrapolation, the numerical put option value could be solved quickly as long as the boundary is known. Eq. (7) shows the early exercise premium (EEP) of an American put option:

$$EEP = rK \int_t^T e^{-r(s-t)} \left\{ \int_0^{M^*(s)} \phi(M_t, M, s-t) dM \right\} ds \tag{38}$$

Let $z = \log(M)$, $z^*(t) = \log\big(M^*(t)\big)$ and $k = \log(M^*(s))$, Eq. (38) can be rewritten as:

$$rK \int_t^T e^{-r(s-t)} \left\{ \int_{-\infty}^{k} \phi(z^*(t), z, s-t) dz \right\} ds \tag{39}$$

Eq. (27) gives the multiday characteristic function of the logarithm price at the end of the N^{th} day. $J_N(v)$ is $e^{ivz_0} H^N$, given Z_0, so we can imply the (*s*-*t*) days characteristic function as follows:

$$J_N(v) = e^{iv\log(M^*(t))} H^{s-t} \tag{40}$$

with $N = s - t$ and $Z_0 = log(M^*(t))$. The logarithm (*s*-*t*)-day price density function could be calculated by the Fourier transformation:

$$\phi(z^*(t), z, s-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} J_N(v) dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} e^{ivz^*(t)} H^{s-t} dv \tag{41}$$

Therefore, Eq. (39) can be rewritten into

$$EEP = rK \int_t^T e^{-r(s-t)} \left\{ \int_{-\infty}^{k} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivz} e^{ivz^*(t)} H^{s-t} dv \right) dz \right\} ds \tag{42}$$

After changing the integral order in Eq. (42), we have:

$$rK \int_{-\infty}^{k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} e^{ivz^*(t)} \left\{ \int_t^T e^{-r(s-t)} H^{s-t} ds \right\} dv\, dz. \tag{43}$$

After defining $J'(v)$ as:

$$J'(v) \equiv e^{ivz^*(t)} \left\{ \int_t^T e^{-r(s-t)} H^{s-t} ds \right\}$$

$$= e^{ivz^*(t)} \frac{1 - e^{-r(T-t)} H^{(T-t)}}{r - \log(H)} = \frac{J_0(v, z^*(t), t_0) - e^{-r(T-t)} J_N(v, z^*(t), T-t)}{r - \log(H)} \tag{44}$$

where $N = T - t$, we could simplify Eq. (43) as follows

$$EEP = rK \int_{-\infty}^{k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} J'(v) dv\, dz. \tag{45}$$

It is clear that the inner integral of Eq. (45) is also a Fourier transform, which means there exists $q'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} J'(v) dv$ such that $J'(v) = \int_{-\infty}^{\infty} e^{ivz} q'(z) dz$.

Then Eq. (45) could be abbreviated as:

$$EEP = rK \int_{-\infty}^{k} q'(z) dz \tag{46}$$

Finally, we let

$$G'(k) \equiv \int_{-\infty}^{k} q'(z) dz = \int_{-\infty}^{k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} J'(v) dv\, dz \tag{47}$$

We define $g'(k) \equiv e^{-\alpha k} G'(k)$ and $\psi'(v) \equiv \int_{-\infty}^{\infty} e^{ivk} g'(k) dk$. After applying an inverse Fourier transformation, we have

$$G'(k) = \frac{e^{\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi'(v) dv. \tag{48}$$

After changing the integral order, we have

$$\psi'(v) = \int_{-\infty}^{\infty} e^{ivk} \int_{-\infty}^{k} e^{-\alpha k} q'(z) dz\, dk$$

$$= \int_{-\infty}^{\infty} \int_{z}^{\infty} e^{(iv-\alpha)k} dk\, q'(z) dz = \int_{-\infty}^{\infty} \frac{e^{i(v+i\alpha)z}}{\alpha - iv} q'(z) dz = \frac{J'_v(v+i\alpha)}{\alpha - iv}\,. \tag{49}$$

With $\alpha > 1$, we have the EEP

$$EEP = rK \frac{e^{\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi'(v) dv, \tag{50}$$

With $\alpha > 1$, the approximation for EEP in Eq. (50) using FFT is given by

$$EEP(k_u) = rK \frac{exp(\widehat{\alpha} k_u)}{\pi} \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j} \psi'(v_j) \frac{\xi}{3} \left[3 + (-1)^j - \varrho_{j-1}\right], \tag{51}$$

where $v_j = \xi(j-1)$, $k_u = -b + \lambda(u-1)$ for $u = 1, 2, \cdots, M$, $b = M\lambda/2$, $\lambda\xi = 2\pi/M$, and ϱ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise.

III. Numerical Results and Findings

3.1 Numerical results

In this section, we discuss the influence of discrete dividends on distribution in daily price limit markets by comparing the results of the proposed numerical solutions with simulations. Table 1 shows the solutions of Guo and Chang (2020) (denoted by GC) and our proposed three-point Richardson extrapolation solutions of the Chang et al. (2016) framework (denoted by RE) are consistent with the results of Monte Carlo simulations (denoted by MC) for European options and those of the least square Monte Carlo simulations (denoted by LSMC) for American options on stocks without dividends in daily price limit markets.[11] The differences between the analytic solutions and MC are quite small. As for the computation time in the framework of Chang et al. (2016), our extended solution seems not to increase with the time to maturity. Our method has a great advantage in time consumption. For example, Table 1 shows that the computation time of our numerical solution is much less than the MC and LSMC. The computation time of our solution may consume more time for American put options, but it seems not to increase with the time to maturity and is apparently between six and seven seconds. However, the computation time of LSMC

[11] According to Hull (2000), American calls on stocks without dividends have no reason to be early exercised and could be treated as European ones.

often quickly increases as the time to maturity increases. The comparison of the computation time of these two methods shows that the Richardson extrapolation is quite accurate and effective for the EEP in markets with daily price limits.

Tables 2 and 3 show the results of our proposed method in the Haug et al. (2003) framework (denoted by HF) when there is a discrete dividend D distributed at the half time of maturity $T/2$. Our numerical solutions denoted by HF are consistent with the MC. The differences between our analytic solutions and MC are less than 0.8% except for some out-of-money European calls. Comparing Tables 1 and 2, we note that discrete dividends actually decrease call option values but increase put option values. Put option values seem to increase with amplitude of D but call option values seem to reduce much less than D. There is a similar phenomenon for American options.

As for American put options on stocks with discrete dividends, the possibility to early exercise in markets with daily price limits and ex-dividends would affect option values. For illustration, we consider an American put option with time to maturity T=24 days, and time to ex-dividend $T/2$=12 days in markets with a daily price limit of 10%. Figure 1 shows that we may not find the early exercise boundary prior to the ex-dividend day. We note that even without the dividend (D=0), the stock price in markets with daily price limits could not fall below the dotted red line to reach the early exercise boundary (the solid red line) before the eighth day. The reason is that the stock price is restrained by the mechanism of daily price limits to trigger the behavior of early exercise before the eighth day.

Taking the event of ex-dividend into account, the dotted green line (denoted by D=5) and the solid blue line (denoted by D=10) in Figure 1 are both even higher than the red line (denoted by the no-dividend case D=0) illustrating that ex-dividend actually increases the possibility to early exercise after the ex-dividend day. The early exercise boundary heavily depends on the discrete dividend D. If it was possible to exercise early before the ex-dividend, the early exercise boundary for D=5 or D=10 would be much lower than that for the no-dividend case. However, as we mentioned above, the stock price could be restrained by the mechanism of daily price limits to trigger the behavior of early exercise prior to the ex-dividend day. If it is not possible to exercise early prior to the ex-dividend day, we could focus on deriving the early exercise boundary posterior to the ex-dividend day for pricing American puts in markets with daily price limits. Table 3 shows that all the differences between our method and LSMC are smaller than 0.3% for American put options. And all the differences between our method and LSMC are smaller than 0.9% for American call options. Our proposed solutions are in line with the LSMC. Table 3 shows that our method also seems to have a greater advantage in the computation time for American options on stocks with a discrete dividend. For example, when $S_0 = 100$ and $T = 6$, the computation time of an American call respectively required by the HFR and the LSMC are 29.44 and 185.04 (sec.).[12]

3.2 Sensitive Analysis and Findings

Figure 2 shows the relationship between daily price limits (γ) and early exercise boundaries of options on stocks without dividends. A more restrictive daily price limit seems to incur an earlier exercise boundary. However, when the daily price limit is 10% or greater, there seems to be little difference between early exercise boundaries. Figure 3 illustrates the relationship between the interest rate and the early exercise boundary for options on stocks with discrete dividends and shows that the interest rate could have a great influence on the early exercise boundary. When interest rates are positive, the higher the interest rate is, the earlier the put option could be exercised. On the contrary, when interest rates are negative, put options could be exercised earlier as interest rates become more negative. Figure 4 shows that the early exercise boundaries in the case of the negative interest rates. Compared to Figure 1, we notice

[12] As for options on stocks with two discrete dividends, Table A1 in the appendix shows that our framework also works well and the differences between the results of our method and the simulation are less than 1.3% except for some out-of-money European call. Therefore, our method could correctly value options on stocks even with several discrete dividends. However, it may be more time-consuming than the simulation method when there is more than one discrete dividend because of triple numerical integrals.

that, in the case of negative interest rates, all the boundaries commonly go down and also seem to heavily depend on the discrete dividend *D*. Figure 5 shows that daily price limits force American put options on stocks with discrete dividends to be exercised earlier when the daily price limits become more restrictive. Although the daily price limits greater than 10% may have little influence on the early exercise behavior of options on stocks without dividends (see Figure 2), we find that a narrower daily price limit moves up the early exercise boundary of options on stocks with discrete dividends. Figures 6 and 7 further indicate that the early exercise boundary for puts on stocks with a discrete dividend is primarily affected by the lower limit instead of the upper limit.

Figure 8 compares the difference between the daily-price limit model and the Black-Scholes (denoted by BS) approximating model whose volatility was implied by option values of Guo and Chang (2020). The shapes of both distributions are quite similar to each other. Because of the simplicity of the BS density function, it could be reasonable to approximate the probability density of stock prices in markets with 10% (or greater) daily price limits by the density function of the BS model in order to accelerate the computation when there is more than one discrete dividend.

Tables 4 and 5 exhibit our results from the approximation of the stock density function in markets with daily price limits by the density function of the BS model.

With this technique, the complicated calculation of the FFT for the probability density could be circumvented. All of the differences between the simulation and HF (or HFR) are smaller than 1% except for OTM calls; the time consumption is also much less than both the MC and the LSMC.

IV. Conclusion

In the valuation of American options, the derivation of the early exercise boundary often involves a recursive and numerical computation and poses practical problems. We find that the three-point Richardson extrapolation improves the computation efficiency of the EEP and extends this new efficient scheme to pricing options on stocks paying discrete dividends in markets with daily price limits. To the best of our knowledge, no study has yet applied this methodology for equity options on stocks paying discrete dividends in markets with daily price limits.

We first extend the intraday density function of Guo and Chang (2020) to a multi-day density function for stocks in markets with daily price limits. Then, we apply our multi-day density function using the framework of Haug et al. (2003) to value European options on stocks paying discrete dividends. Moreover, we build an efficient formula and take advantage of FFT to quickly calculate the EEP in markets with daily price limits. We also adopt the three-point Richardson extrapolation to accelerate the computation of American options. The accuracy of our proposed method is further verified by simulations. The mechanism of the daily-price limits would force American put options to be barely exercised before the ex-dividend. However, daily price limits could make exercising prior to the ex-dividend impossible for short-term put options. We also note that the early exercise boundary goes up when either the positive interest rate or the dividend increases. With fixed dividends, more restrictive daily price limits could force put options to be exercised earlier. In addition, early exercise boundaries could be more sensitive for positive interest rates than for negative interest rates; the lower limit is the primary factor affecting the early exercise boundary for American puts. Finally, we propose an alternative method to approximate the daily-price limit model by the B-S model under some constraints to accelerate the computation.

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Table 1. Options on Stocks without Dividends in Markets with Daily Price Limits

		European Call					European Put					American Put				
S_0	T	GC	Time	MC	Time	Diff	GC	Time	MC	Time	Diff	RE	Time	LSMC	Time	Diff
90	6	0.88	0.06	0.88(0.007)	3.10	0.52%	10.86	0.06	10.88(0.005)	3.10	0.15%	10.88	7.14	10.86(0.013)	195.24	0.23%
	12	2.09	0.06	2.09(0.018)	6.63	0.08%	12.04	0.06	12.08(0.011)	6.63	0.31%	12.06	6.36	12.05(0.017)	684.57	0.07%
	24	4.11	0.06	4.10(0.015)	16.73	0.30%	14.02	0.06	14.08(0.007)	16.72	0.43%	14.10	7.45	14.04(0.026)	2945.67	0.46%
100	6	4.30	0.04	4.29(0.012)	3.03	0.23%	4.28	0.04	4.29(0.011)	3.03	0.32%	4.30	6.99	4.28(0.008)	189.60	0.45%
	12	6.08	0.04	6.06(0.021)	6.61	0.34%	6.04	0.04	6.06(0.015)	6.61	0.43%	6.05	6.34	6.05(0.011)	686.61	0.01%
	24	8.60	0.04	8.56(0.026)	16.32	0.47%	8.51	0.04	8.56(0.017)	16.31	0.63%	8.59	7.45	8.54(0.014)	2934.92	0.64%
110	6	11.18	0.03	11.17(0.012)	3.04	0.08%	1.16	0.04	1.17(0.008)	3.04	1.21%	1.18	6.98	1.17(0.007)	177.87	1.14%
	12	12.61	0.03	12.58(0.024)	6.63	0.21%	2.56	0.04	2.58(0.018)	6.63	0.91%	2.57	6.36	2.58(0.015)	643.20	0.32%
	24	14.92	0.03	14.84(0.052)	16.84	0.49%	4.82	0.04	4.85(0.031)	16.84	0.63%	4.90	7.43	4.84(0.024)	2821.80	1.27%

Model parameter specifications: r=1%, K=100, σ =70%, daily price limit γ=10%, T measured in days, and computation time measured in seconds. GC denotes the solution of Guo and Chang (2020). We set ξ=0.1702, $\hat{\alpha}$ =1.1 and use 4096 points in the quadrature. MC denotes the Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations of MC. The absolute value of the different between GC and MC divided by MC is denoted by Diff. RE denotes our proposed three-point Richardson extrapolation solutions of the Chang et al. (2016) framework. LSMC denotes the least square Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote the standard deviations of LSMC. The absolute value of the difference between RE and LSMC divided by LSMC is denoted by Diff.

Table 2. European Options on Stocks with a Discrete Dividend in Markets with Daily Price Limits

		European Call					European Put				
S_0	T	HF	Time	MC	Time	Diff	HF	Time	MC	Time	Diff
90	6	0.10	5.15	0.09(0.003)	3.08	4.50%	20.07	5.16	20.08(0.003)	3.08	0.07%
	12	0.53	8.87	0.54(0.008)	6.67	1.72%	20.50	9.00	20.53(0.005)	6.67	0.15%
	24	1.77	18.31	1.77(0.012)	17.07	0.40%	21.67	18.37	21.74(0.005)	17.07	0.30%
100	6	1.03	5.20	1.02(0.009)	3.07	0.27%	11.01	5.21	11.02(0.006)	3.07	0.08%
	12	2.35	9.49	2.34(0.01)	6.72	0.14%	12.30	9.56	12.33(0.006)	6.72	0.30%
	24	4.51	19.75	4.48(0.024)	16.82	0.76%	14.41	19.83	14.47(0.013)	16.82	0.40%
110	6	4.53	5.27	4.52(0.012)	3.11	0.11%	4.50	5.37	4.52(0.009)	3.11	0.39%
	12	6.40	10.00	6.39(0.035)	6.67	0.17%	6.35	10.13	6.38(0.024)	6.67	0.55%
	24	9.05	21.85	9.00(0.037)	16.96	0.56%	8.95	21.99	8.99(0.023)	16.96	0.54%

Model parameter specifications: r=1%, K=100, σ =70%, D=10, daily price limit γ=10%, T measured in days, and computation time measured in seconds. HF denotes the proposed solution using the FFT in the framework of Haug et al. (2003). We set ξ = 0.1702, $\hat{\alpha}$ =1.1, ΔM=1, and use 4096 points in the quadrature. The dividend D is assumed to be distributed at the time T/2. MC denotes the Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations of MC. The absolute value of the difference between HF and MC divided by MC is denoted by Diff.

Table 3. American Options on Stocks with a Discrete Dividend in Markets with Daily Price Limits

		American Call					American Put				
S_0	T	HFR	Time	LSMC	Time	Diff	HFR	Time	LSMC	Time	Diff
90	6	0.29	31.28	0.29(0.004)	140.64	0.83%	20.07	31.37	20.09(0.006)	195.24	0.11%
	12	1.00	53.92	0.99(0.006)	570.82	0.56%	20.51	49.65	20.51(0.014)	684.57	0.00%
	24	2.47	97.56	2.45(0.019)	2611.76	0.72%	21.70	104.23	21.70(0.017)	2945.67	0.02%
100	6	3.08	29.44	3.08(0.01)	185.04	0.20%	11.00	34.47	10.99(0.012)	189.60	0.06%
	12	4.45	54.58	4.43(0.01)	671.21	0.54%	12.31	58.81	12.30(0.015)	686.61	0.10%
	24	6.50	104.88	6.44(0.023)	2892.98	0.87%	14.45	112.34	14.42(0.022)	2934.92	0.16%
110	6	10.42	26.73	10.41(0.014)	198.37	0.13%	4.50	29.86	4.50(0.012)	177.87	0.10%
	12	11.27	54.42	11.23(0.014)	705.90	0.35%	6.36	58.88	6.36(0.026)	643.20	0.03%
	24	12.92	110.61	12.85(0.022)	3012.30	0.51%	8.98	125.47	8.96(0.026)	2821.80	0.29%

Model parameter specifications: r=1%, K=100, σ =70%, D=10, daily price limit γ=10%, T measured in days, and computation time measured in seconds. HFR denotes the proposed solution using the FFT and Richardson extrapolation in the framework of Haug et al. (2003). We set ξ = 0.1702, $\hat{\alpha}$ = 1.1, ΔM= 1, and use 4096 points in the quadrature. The dividend D is assumed to be distributed at the time T/2. LSMC denotes the least square Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations. The absolute value of the difference between HFR and LSMC divided by LSMC is denoted by Diff.

Table 4. The Efficiency of BS Approximations for European Options on Stocks with a Discrete Dividend in Markets with Daily Price Limits

			Call					Put				
S_0	T	$\hat{\sigma}$	HF	Time	MC	Time	Diff	HF	Time	MC	Time	Diff
90	6	69.47%	0.09	3.15	0.09(0.003)	3.08	3.84%	20.07	3.16	20.08(0.003)	3.08	0.07%
	12	69.64%	0.54	3.95	0.54(0.008)	6.67	0.39%	20.49	3.98	20.53(0.005)	6.67	0.18%
	24	69.67%	1.76	5.05	1.77(0.012)	17.07	0.36%	21.66	5.11	21.74(0.005)	17.07	0.36%
100	6	69.86%	1.03	3.06	1.02(0.009)	3.07	0.25%	11.00	3.11	11.02(0.006)	3.07	0.16%
	12	69.75%	2.34	3.84	2.34(0.010)	6.72	0.30%	12.29	3.88	12.33(0.006)	6.72	0.38%
	24	69.69%	4.49	5.03	4.48(0.024)	16.81	0.28%	14.39	5.07	14.47(0.013)	16.81	0.55%
110	6	69.68%	4.51	2.93	4.52(0.012)	3.11	0.26%	4.48	2.97	4.52(0.009)	3.11	0.76%
	12	69.74%	6.38	3.92	6.39(0.035)	6.67	0.09%	6.33	3.94	6.38(0.024)	6.67	0.81%
	24	69.71%	9.02	5.24	9.00(0.037)	16.95	0.22%	8.92	5.28	8.99(0.023)	16.95	0.88%

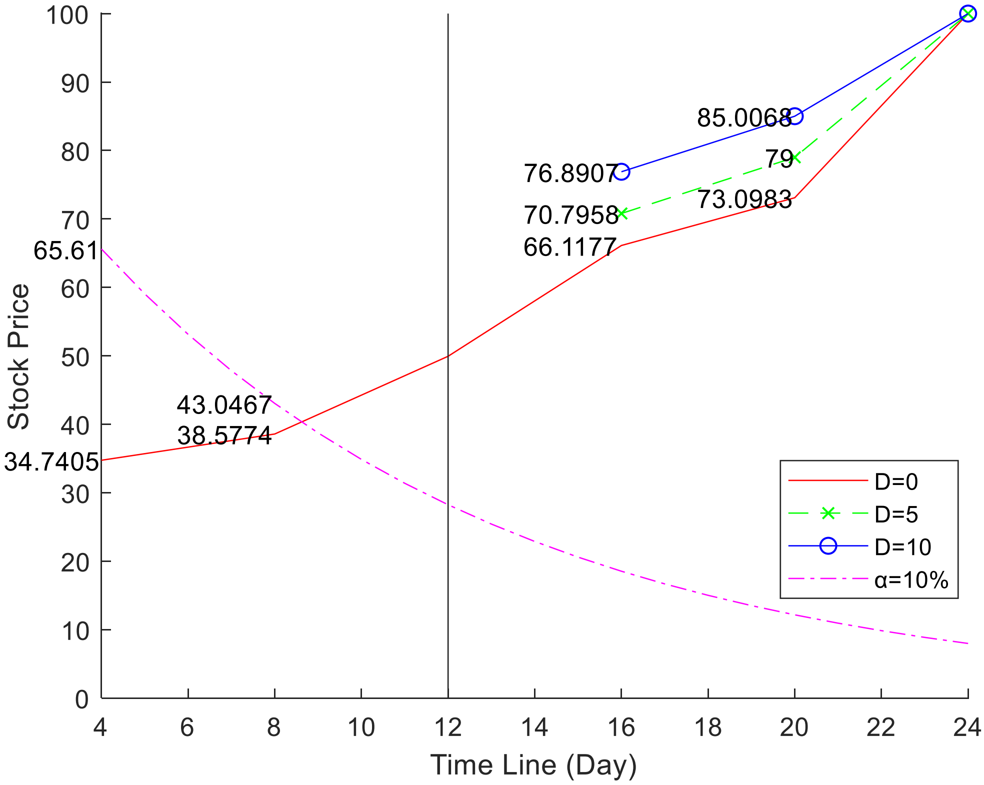
Model parameter specifications: r=1%, K=100, σ =70%, D=10, daily price limit γ=10%, T measured in days, and computation time measured in seconds. HF denotes the proposed solution using the FFT in the framework of Haug et al. (2003). We set ξ =0.1702, $\hat{\alpha}$ =1.1, ΔM = 1, and use 4096 points in the quadrature. The $\hat{\sigma}$ denotes the implied volatility from the daily-price limit European call option value with the BS model. The dividend D is assumed to be distributed at the time $T/2$. MC denotes the Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations of MC. The absolute value of the difference between HF and MC divided by MC is denoted by Diff.

Table 5. The Efficiency of BS Approximations for American Options on Stocks with a Discrete Dividend in Markets with Daily Price Limits

			American Call					American Put				
S_0	T	$\hat{\sigma}$	HFR	Time	LSMC	Time	Diff	HFR	Time	LSMC	Time	Diff
90	6	69.47%	0.29	16.06	0.29(0.004)	140.64	0.25%	20.06	32.10	20.09(0.006)	195.24	0.14%
	12	69.64%	0.99	19.19	0.99(0.006)	570.82	0.47%	20.50	34.73	20.51(0.014)	684.57	0.04%
	24	69.67%	2.45	24.50	2.45(0.019)	2611.76	0.05%	21.69	39.14	21.70(0.017)	2945.67	0.08%
100	6	69.86%	3.07	11.60	3.08(0.010)	185.04	0.11%	11.00	33.93	10.99(0.012)	189.60	0.03%
	12	69.75%	4.43	16.09	4.43(0.010)	671.22	0.07%	12.30	37.54	12.30(0.015)	686.61	0.04%
	24	69.69%	6.47	21.43	6.44(0.023)	2892.98	0.40%	14.44	43.16	14.42(0.022)	2934.93	0.08%
110	6	69.68%	10.40	7.99	10.41(0.014)	198.37	0.04%	4.49	35.38	4.50(0.012)	177.88	0.36%
	12	69.74%	11.24	12.47	11.23(0.014)	705.90	0.14%	6.36	39.36	6.36(0.026)	643.20	0.10%
	24	69.71%	12.88	18.65	12.85(0.022)	3012.30	0.19%	8.97	45.76	8.96(0.026)	2821.80	0.17%

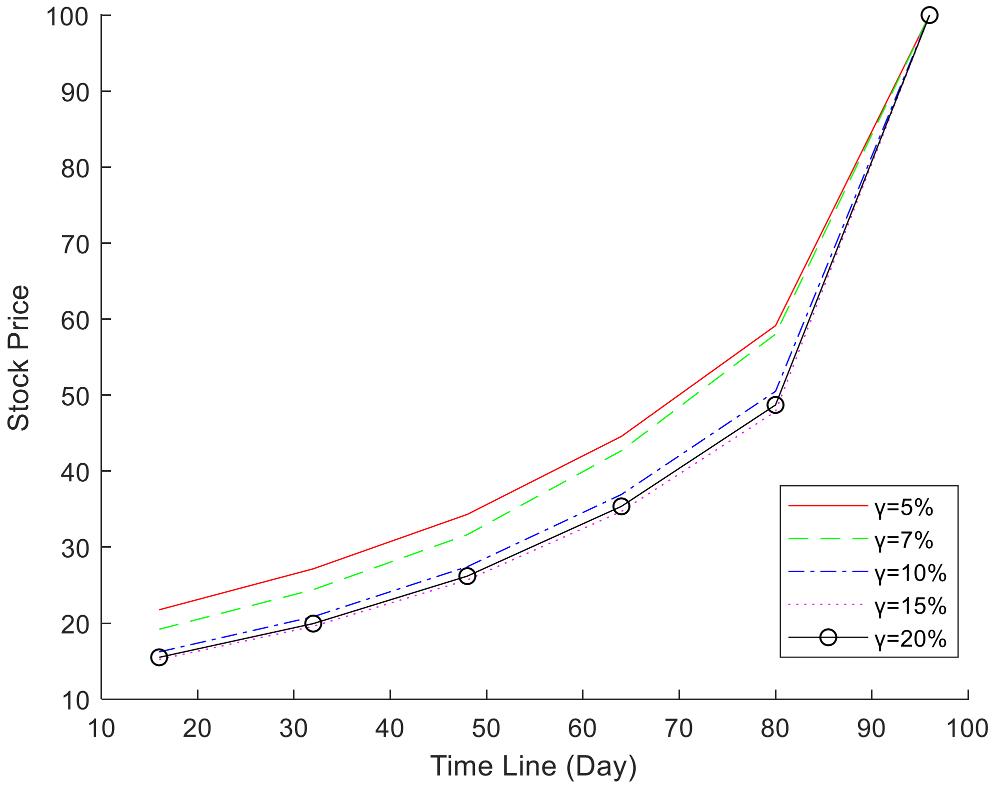
Model parameter specifications: r=1%, K=100, σ =70%, D=10, daily price limit γ=10%, T measured in days, and computation time measured in seconds. HFR denotes the proposed solution using the FFT and Richardson extrapolation in the framework of Haug et al. (2003). We set ξ = 0.1702, $\hat{\alpha}$ =1.1 , ΔM=1, and use 4096 points in the quadrature. The $\hat{\sigma}$ denotes the implied volatility from the daily-price limit European call option value with the BS model. The dividend D is assumed to be distributed at the time $T/2$. LSMC denotes the least square Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations. The absolute value of the difference between HFR and LSMC divided by LSMC is denoted by Diff.

Figure 1. Early Exercise Boundary for Puts on Stocks with Discrete Dividends in Markets with Daily Price Limits



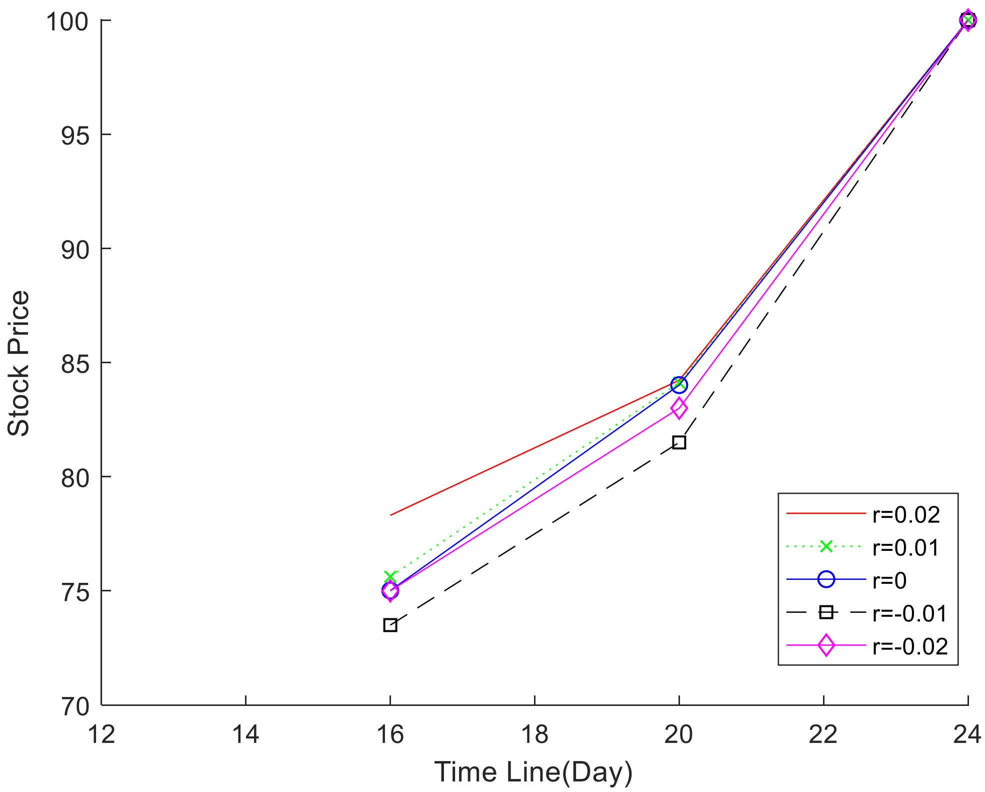
Model parameter specifications: S_0=K=100, r =10%, σ=70%, time to maturity N=24 days, the ex-dividend day T_D=12, and daily-price limit γ =10%. The line denoted by α =10% is the lower boundary of the stock price in markets with daily price limits.

Figure 2. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks without Dividends to Daily Price Limits



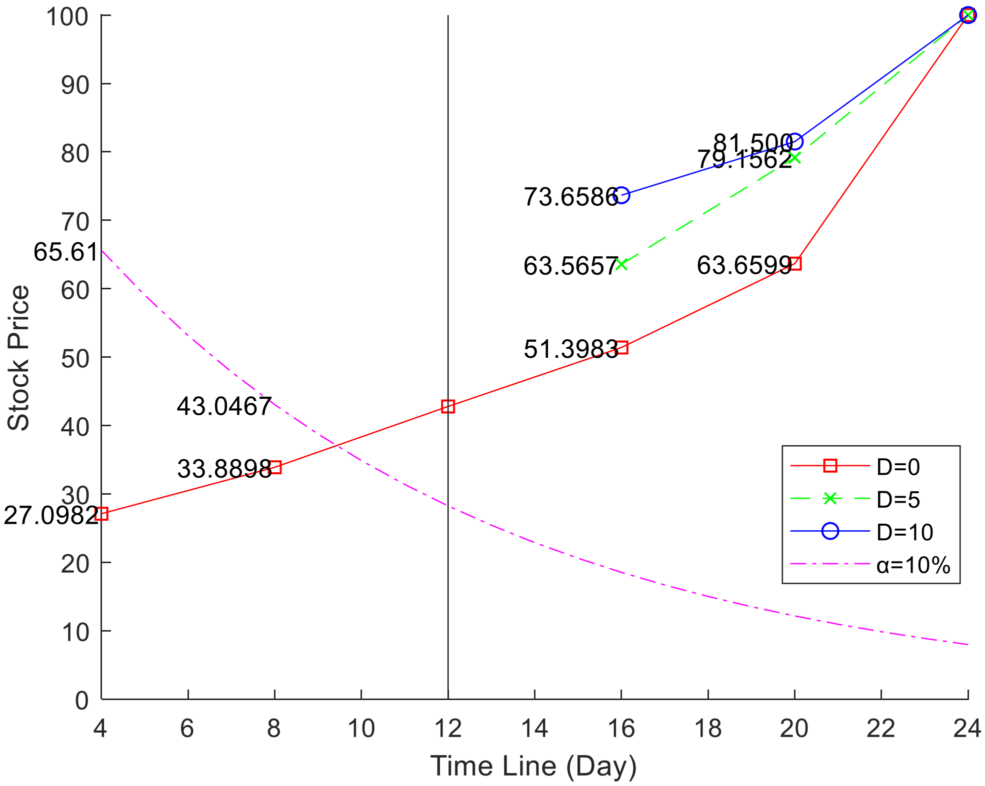
Model parameter specifications: S_0=K=100, r =10%, σ=70%, time to maturity N=96 days, dividend D=0, and daily-price limit denoted by γ.

Figure 3. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks with a Discrete Dividend to Interest Rates (Dividend D Fixed at 10 dollars)



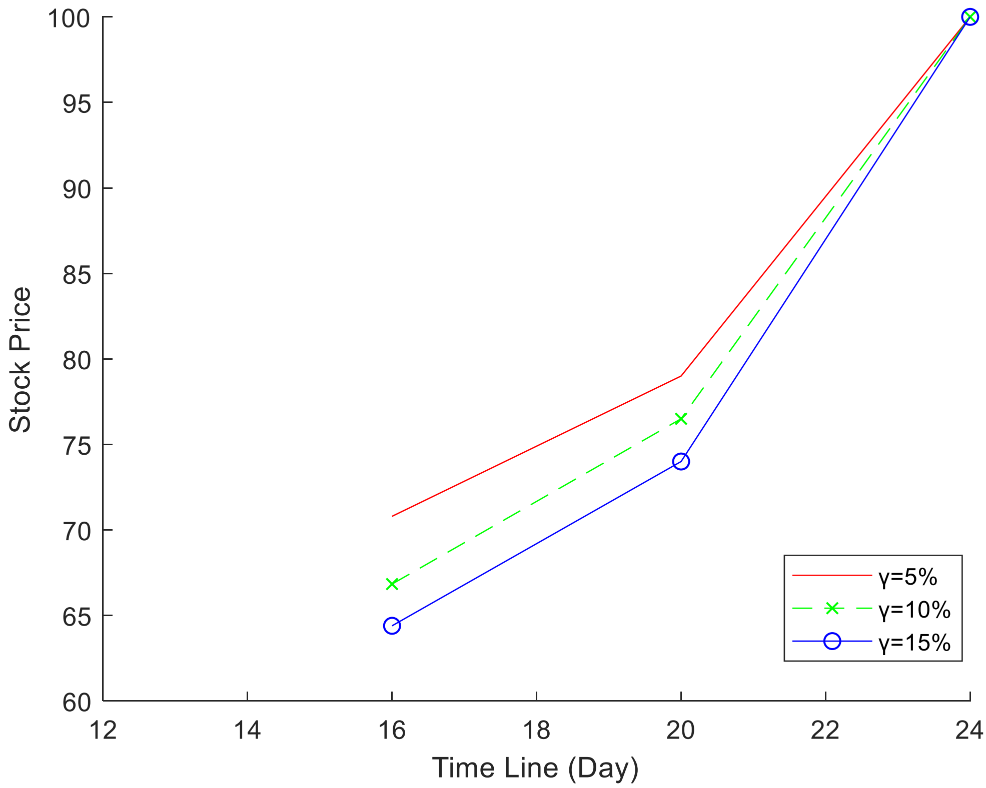
Model parameter specifications: $S_0 = K = 100$, σ =70%, time to maturity N =24 days, dividend D =10, ex-dividend day T_D =12, and daily-price limit γ =10%.

Figure 4. A Sensitive Analysis of Early Exercise Boundaries for Puts to Discrete Dividends with Negative Interest Rates



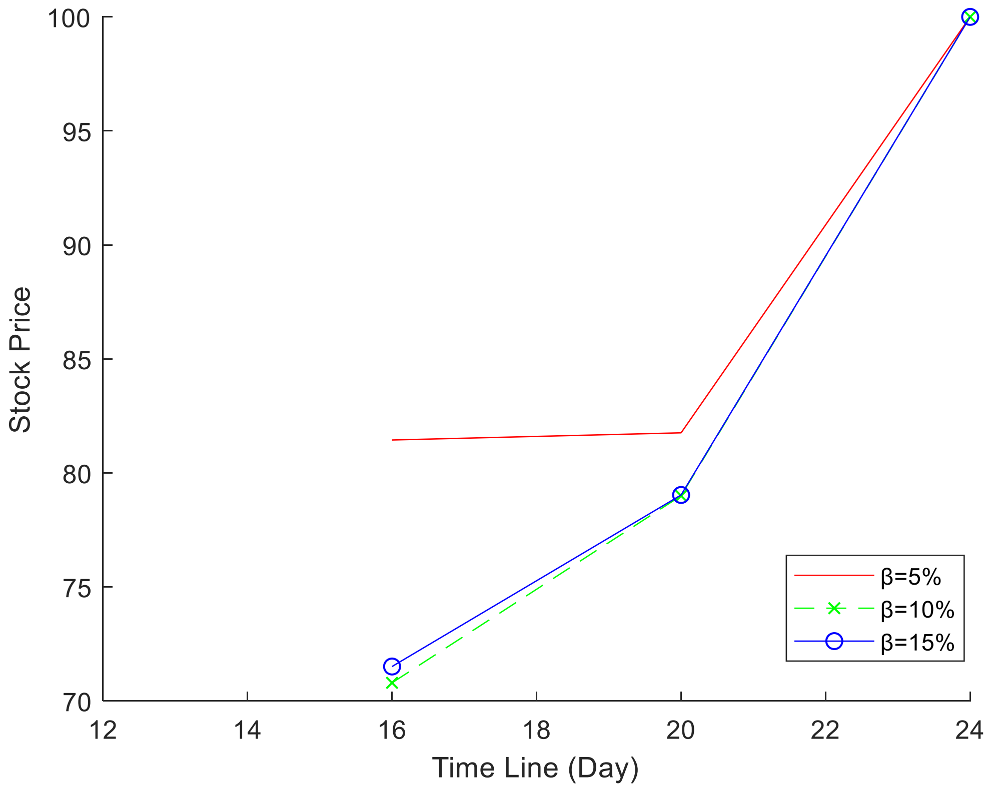
Model parameter specifications: S_0=K=100, r = -10%, σ=70%, time to maturity N=24 days, ex-dividend day T_D=12, and daily-price limit γ=10%. The line denoted by α =10% is the lower boundary of the stock price in markets with daily price limits.

Figure 5. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks with a Discrete Dividend to Daily Price Limits



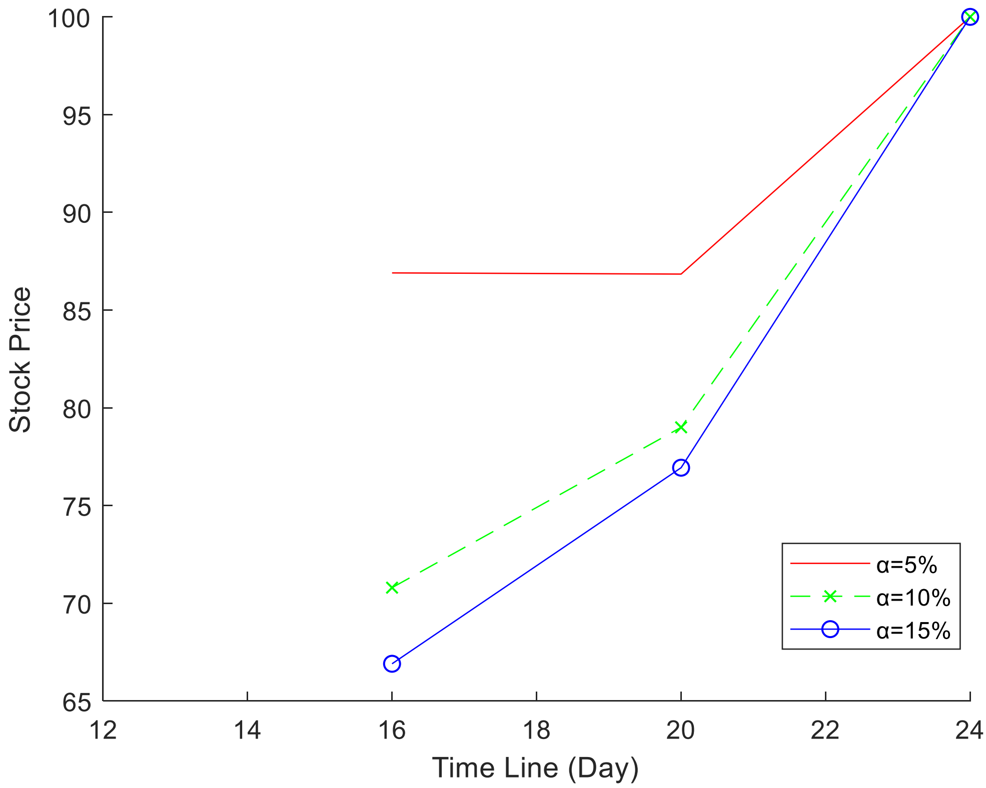
Model parameter specifications: $S_0 = K = 100$, $r = 10\%$, $\sigma = 70\%$, time to maturity $N = 24$ days, dividend $D = 5$, ex-dividend day $T_D = 12$, and daily-price limit denoted by γ.

Figure 6. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks with a Discrete Dividend to Upper Price Limits (Dividend *D* Fixed at 5)



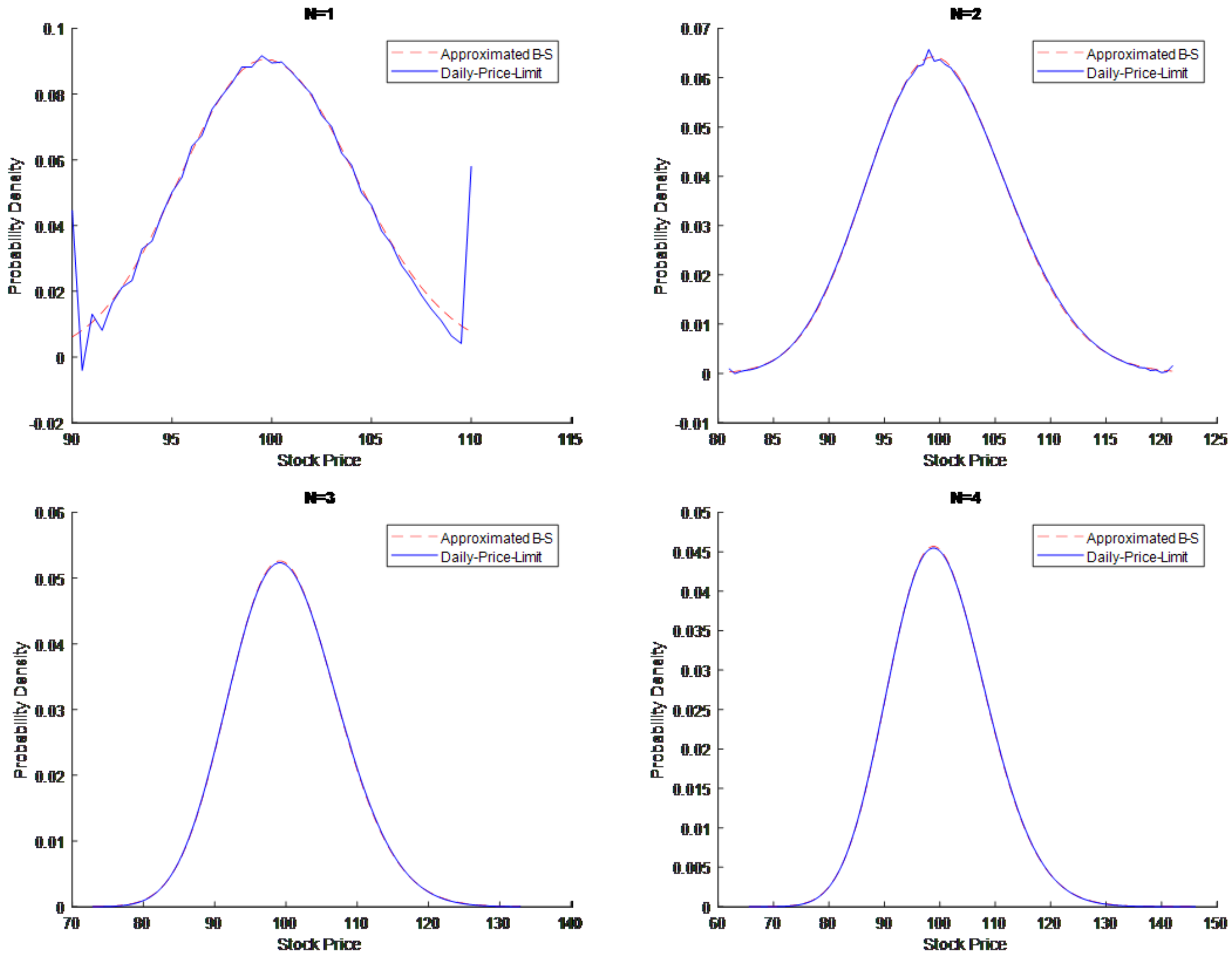
Model parameter specifications: $S_0 = K = 100$, $r = 10\%$, $\sigma = 70\%$, time to maturity $N = 24$ days, dividend D=5, ex-dividend day T_D=12, the lower daily-price limit $\alpha = 10\%$, and the upper daily-price limit denoted by β.

Figure 7. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks with a Discrete Dividend to Lower Price Limits (Dividend D Fixed at 5)



Model parameter specifications: $S_0 = K = 100$, $r = 10\%$, $\sigma = 70\%$, time to maturity $N = 24$ days, dividend D=5, ex-dividend day T_D=12, the upper daily-price limit $\beta = 10\%$, and the lower daily-price limit denoted by α.

Figure 8. A Comparison of Stock Density Function in Markets with Daily Price Limits to its BS Approximation



Model parameter specifications: S_0 =K =100, r =10%, σ =70%, daily-price limits γ =10%, and the Black-Scholes model denoted by BS.

Appendix

Table A1. Options on Stocks with Two Discrete Dividends in Markets with Daily Price Limits

	European Call					European Put				
S_0	HF	Time	MC	Time	Diff	HF	Time	MC	Time	Diff
90	0.04	242.46	0.03(0.001)	4.68	9.65%	30.00	248.57	30.02(0.004)	4.68	0.07%
100	0.37	284.12	0.37(0.007)	4.99	1.22%	20.34	292.45	20.36(0.003)	4.99	0.13%
110	1.92	331.12	1.92(0.014)	4.75	0.07%	11.88	342.08	11.91(0.009)	4.75	0.25%
	American Call					American Put				
S_0	HFR	Time	LSMC	Time	Diff	HFR	Time	LSMC	Time	Diff
90	0.29	9984.21	0.29(0.0035)	281.11	1.04%	30.00	1846.08	30.03(0.006)	816.00	0.11%
100	3.08	7691.37	3.08(0.0093)	352.22	0.18%	20.33	2194.96	20.36(0.005)	768.26	0.15%
110	10.43	5502.17	10.40(0.0114)	359.98	0.25%	11.88	2545.16	11.90(0.004)	782.66	0.19%

Model parameter specifications: r=1%, K=100, σ =70%, T=9 days, $D1$=10, $D2$=10, daily price limit γ=10%, and computation time measured in seconds. HF denotes the proposed solution using the FFT in the framework of Haug et al. (2003). HFR denotes the proposed solution using the FFT and Richardson extrapolation in the framework of Haug et al. (2003). We set ξ =0.1702, $\hat{\alpha}$ =1.1, ΔM =1, and use 4096 points in the quadrature. The dividends D_1 and D_2 are assumed to be distributed at the time T=3 and T=6, respectively. MC denotes the Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. LSMC denotes the least square Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. We use the absolute value of the difference between HF and MC divided by MC to calculate the Diff of European options. We use the absolute value of the difference between HFR and LSMC divided by LSMC to calculate the Diff of American options. Numbers in brackets denote the standard deviations of MC or LSMC.